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Chen-Burr XII Distribution

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ABSTRACT

Survival analysis focuses on modeling the time until an event occurs, where the hazard function determines the flexibility of distributions for modeling lifetime data. The Chen distribution, defined by two positive parameters, can represent monotonically increasing and bathtub-shaped hazard functions. However, it cannot accommodate monotonically decreasing and unimodal (upside-down bathtub) hazard patterns, which are frequently encountered in failure risk and medical survival data. This study aims to construct a more flexible distribution capable of accommodating four hazard shapes: increasing, decreasing, unimodal, and bathtub-shaped using the Transformed-Transformer $(T - X)$ method. To address this limitation, the Burr Type XII distribution is used as the transformer distribution due to its ability to capture decreasing and unimodal hazard patterns, while the Chen distribution acts as the transformed distribution. The resulting Chen-Burr XII distribution has four positive parameters (λ, β, c, k). Characteristics discussed in this study are the probability density function, cumulative distribution function, survival function, hazard function, and r-th moment. The maximum likelihood estimation approach is used to estimate parameter values. The Chen-Burr XII distribution is applied to survival data of gastric cancer patients' waiting time to death, exhibiting a unimodal hazard function. Model performance is evaluated using the Kolmogorov-Smirnov test and Akaike's Information Criterion. The results show that the Chen-Burr XII distribution models data with a lower AIC value compared to the Chen distribution, offering a more flexible alternative for lifetime data with diverse hazard patterns.

Keywords: Lifetime Data, Maximum Likelihood Estimation, Survival Analysis, Transformed-Transformer $(T - X)$ Method, Unimodal Hazard Function.

1. Introduction

Survival analysis is a branch of data analysis and statistical modeling that focuses on observing the time until a particular event occurs. Survival analysis aims to explore time-related data to discover patterns and trends in phenomena and to examine the influence of various factors on the risk of an event occurring. In general, distribution characteristics can be identified through the probability density function or cumulative distribution function. However, in survival analysis that examines lifetime data, the hazard function (force of mortality) is considered an important indicator of the distribution's flexibility in modeling event risk. Therefore, the hazard function is more informative in survival analysis than the survival function because of its ability to capture various risk patterns so that it can describe the dynamics of events that occur over time. In addition, survival data is often incomplete, either in the form of censored data or truncated data. In this situation, the data characteristics cannot be accurately described solely by the probability distribution expressed by the probability density function (PDF). Instead, a hazard function is needed to more accurately describe the characteristics of the survival data. Therefore, developing a distribution with more flexible hazard functions is crucial in modeling survival data.

In survival analysis, various forms of hazard functions are often used as modeling assumptions, such as increasing, decreasing, upside-down bathtub, and bathtub-shaped. The Chen distribution was introduced in 2000 as a distribution with two positive parameters (λ, β) is able to represent both monotonically increasing and bathtub-shaped hazard functions and is valid for all positive values of $x > 0$. The Chen distribution has the characteristic that the hazard function can be increasing when $\beta \geq 1$ and bathtub-shaped $\beta < 1$ [1]. Although the Chen distribution has the flexibility to represent both increasing and bathtub-shaped hazards, this distribution is not yet able to model the unimodal hazard form that is commonly found in survival data analysis, such as in the case of the risk of death. Therefore, it is necessary to extend the modifications to the Chen distribution to capture hazard patterns in modeling various forms of lifetime data.

Several methods, such as the summation of several random variables and the transformation of random variables by scalar multiplication are used. The XTG distribution is an example of the results of the expansion of the Chen distribution, M. Xie, Y. Tang, and T. N. Goh, by using the random variable transformation method against scalar multiplication so that the addition of scale parameters produces an increasing and bathtub-shaped hazard form [2]. Despite improved flexibility, the hazard shapes remain limited to increasing and bathtub-shaped patterns. To overcome this limitation, a method is needed that not only increases the complexity of the distribution through parameter addition but also overcomes the Chen distribution's shortcomings in accommodating decreasing and unimodal hazard shapes.

The Transformed-Transformer $(T - X)$ method, first introduced by Alzaatreh in 2013, offers an alternative solution. In this method, T represents a random variable representing the transformed distribution, while X represents the transformer distribution. The Burr Type XII distribution is a distribution frequently used in modeling various fields, such as reliability analysis, life testing, survival analysis, actuarial science, and economics [4]. The Burr distribution has a monotonically decreasing hazard form when $k > 0, 0 < c \leq 1$ and an upside-down bathtub form, $k > 0, c > 1$ making it a suitable alternative to increase the flexibility of the Chen distribution. This method yields a new distribution, known as the Chen-Burr XII distribution, with four positive parameters $\lambda, \beta, c,$ and k.

The structure of this paper is as follows: the theoretical framework for the proposed distribution is presented in Section 2. The construction of the Chen-Burr XII distribution using the Transformed-Transformer $(T - X)$ method and analysis of its statistical characteristics, including the cumulative distribution function, probability density function, survival function, hazard function, r-th moment, and parameter estimation using the Maximum Likelihood Estimation (MLE) method, are all covered in Section 3. The results and discussion of implementing the proposed distribution on lifetime data are presented in Section 4. The Total Time-On-Test (TTT) plot is used to assess the adequacy of the hazard function before the Kolmogorov-Smirnov test is used to test for goodness of fit. The model's performance is compared with the Chen and Burr Type XII distributions using the Kolmogorov-Smirnov test and the Akaike Information Criterion (AIC). It is also evaluated by applying it to gastric cancer survival data with a unimodal hazard pattern. Lastly, a summary of the study's conclusions and contributions is given in Section 5.

2. Theoretical Framework

2.1. Transformed-Transformer Method

This method was introduced by Alzaatreh in 2013 with the aim of increasing flexibility by using PDF as a generator and extending the model by adding parameters to model various data types. The basic concept of this method involves two random variables: a transformed variable, which is a random variable from the distribution to be formed, and a transformer variable, or the variable used as the basis for the transformation [3].

For example:

1. $W(F(x)) \in [a, b]$
2. $W(F(x))$ differentiable and monotone non-decreasing

(1)

3. $W(F(x)) -> a$ if $x -> -\infty$ and $W(F(x)) -> b$ if $x -> \infty$

The family of distributions generated by this method is called a "distribution family," and the new distribution function can be defined as

$$G(x) = \int_{a}^{W(F(x))} r(t)\, dt \tag{2}$$

The selection of functions that depend on specific distribution value spaces will determine the resulting new family of distributions, as follows:

1. When the value space is restricted to [a, b], $W(F(x))$ can be defined as $F(x)$. If it is assumed to be restricted to [0,1], then the resulting distributions include the Uniform$(0,1)$, Beta, Kumaraswamy, and Generalized Beta distributions.
2. When the value space is $[a, \infty]$, $a \geq 0$, $W(F(x))$ can be defined as $-\log(1 - F(x))$ and $\frac{F(x)}{1-F(x)}$.
3. When the value space is $(-\infty, \infty)$, $W(F(x))$ can be defined as $\log[-\log(1 - F(x))]$ and $\log\left(\frac{F(x)}{1-F(x)}\right)$.

2.2. Chen Distribution

The Chen distribution with parameters λ and β or mathematically written, $X \sim \text{Chen}(\lambda, \beta)$ has a distribution function for $x > 0$, parameters $\lambda, \beta > 0$ as follows.

$$F(x) = 1 - \exp\left[\lambda\left(1 - e^{x^{\beta}}\right)\right] \tag{3}$$

The probability density function of the Chen distribution for $x > 0$ the parameters $\lambda, \beta > 0$ is as follows.

$$f(x) = \lambda\beta x^{\beta-1} \exp\left[x^{\beta} + \lambda\left(1 - e^{x^{\beta}}\right)\right] \tag{4}$$

The survival function and hazard function of the Chen distribution for $x > 0$ the parameters $\lambda, \beta > 0$ are provided, respectively, by:

$$S(x) = \exp\left[\lambda\left(1 - e^{x^{\beta}}\right)\right] \tag{5}$$

and

$$h(x) = \lambda\beta x^{\beta-1} e^{x^{\beta}} \tag{6}$$

The hazard function of the Chen distribution is bathtub-shaped when $\beta < 1$ and increasing when $\beta \geq 1$.

2.3. Burr Type XII Distribution

The Burr Type XII distribution is derived from the Burr Type III distribution by replacing x with $x - 1$. Of the twelve types of distribution functions, the Burr Type XII is the simplest and most frequently used in statistical modeling in various fields. According to [5], Burr Type XII distribution has gained attention compared to other Burr distributions due to its ability to model various applications, such as the failure of insulation pipes [6] and the waiting time to relapse in leukemia patients after transplantation [7]. Therefore, the Burr Type XII distribution is considered a common parametric distribution model and is frequently used in reliability analysis and survival data and is known as the Burr distribution. The Burr distribution has two positive shape parameters, k and c. This results in the Burr distribution's hazard function being decreasing or unimodal, making it frequently used in survival analysis modeling [4]. The following is the cumulative distribution function of the Burr XII distribution for $x > 0$ and $k, c > 0$.

$$F(x) = 1 - (1 + x^c)^{-k} \tag{7}$$

The probability density function of the Burr XII distribution for $x > 0$ the parameters $c, k > 0$ is as follows.

$$f(x) = kcx^{c-1}(1 + x^c)^{-(k+1)} \tag{8}$$

The survival function and hazard function of the Burr XII distribution for $x > 0$ the parameters $c, k > 0$ are provided, respectively, by:

$$S(x) = (1 + x^c)^{-k} \tag{9}$$

And

$$h(x) = \frac{kcx^{c-1}}{(1 + x^c)} \tag{10}$$

The hazard function of the Burr distribution with parameters $k > 0, c > 1$ takes the form of an upside-down bathtub for time and is monotonically decreasing when the parameter values satisfy $k > 0$ and $0 < c \leq 1$. In statistical applications, hazard functions exhibiting monotonically decreasing and upside-down bathtub shapes are commonly used in survival analysis modeling.

2.4. Total Time-On-Test (TTT)

The Total Time-On-Test statistic, commonly known as TTT, was introduced by Epstein and Sobel in 1953 to interpret the properties of Exponential distribution. Researchers have found many other uses for TTT for various applications, such as model identification, the basis for characterizing survival distributions, and others. [8] Barlow & Campo first used TTT to identify non-monotonic hazard forms, such as decreasing, increasing, and constant. Subsequently, Aarset [9] expanded this approach by showing that TTT can also be used to identify more complex hazard forms, including unimodal and bathtub hazards by looking at the resulting TTT curve. The statistic that states the number of times for a test of n random samples to experience an event is denoted as T_n. Suppose the random variable X represents the time until the event occurs with $X_1, X_2, \dots, X_n$. Thus, the ordered statistics of random variables $X_1, X_2, \dots, X_n$ can be expressed as $0 \leq X_{(0)} \leq X_{(1)} \leq \dots \leq X_{(n)}$. The statistical value of Total Time-On-Test can be obtained through the following equation.

$$T_n = \sum_{i=1}^{n} X_{(i)}$$

$$T_n = \sum_{i=1}^{n} (n - i + 1)\left(X_{(i)} - X_{(i-1)}\right) \tag{11}$$

The equation can be illustrated in geometric form where the area of the square formed from the vertical length $(n - i + 1)$ and the horizontal length $(X_{(i)} - X_{(i-1)})$ is as follows.

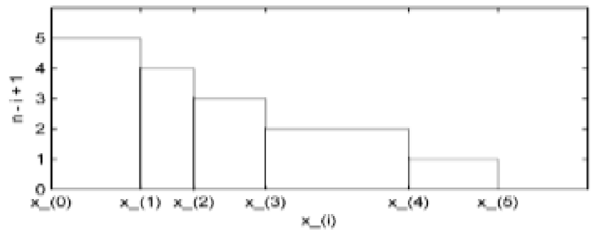


Figure 1. Geometric Illustration Interpretation of T_n

The derived statistics of TTT or successive TTT-statistics denoted by T_i are the total times for testing from n samples until the i-th sequential observation experiences an event, defined as:

$$T_i = \sum_{j=1}^{i} (n - j + 1)\left(X_{(j)} - X_{(j-1)}\right) \tag{12}$$

Based on Equations (11) and (12), it is known that $n = i$, the value of T_i will be the same as the value of T_n which is defined as the scaled TTT-statistics on the interval $[0,1]$ and T_i^* can be denoted as $T\left(\frac{i}{n}\right)$ which is formulated by:

$$T_i^* = \frac{T_i}{T_n}, \qquad i = 1,2, \dots , n \tag{13}$$

By mapping T_i^* on the y-axis and $F_n(X_{(i)}) = \frac{i}{n}$ on the x-axis, the resulting points are connected to form a curve on the (x, y)-plane, which is then referred to as the TTT curve [11]. Through the mathematical relationship between the TTT transformation and the hazard function, the hazard form can be indirectly observed through the resulting TTT curve. The following is an illustration of various hazard forms from data based on the TTT curve.

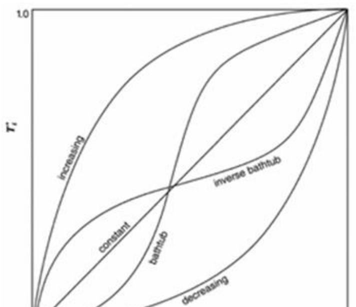


Figure 2. TTT Curve Variations

Based on Figure 2, the TTT curve is a straight line, indicating that the hazard rate is constant. If the TTT curve is concave downwards, or concave, the hazard rate tends to increase monotonically. Conversely, if the TTT curve is concave upwards, or convex, the hazard rate can be said to decrease monotonically. Furthermore, if the TTT curve begins with a concave shape and then changes to a convex shape, the hazard rate has a pattern called unimodal. Conversely, if the TTT curve begins with a convex shape and then changes to a concave shape, the hazard shape resembles a bathtub [11]. These varying forms of hazard rates can then be classified in Table 1 as follows.

Table 1. Classification of TTT Curve Shapes

Hazard Function Shape	TTT Curve Shape (TTT-Plot)
Constant	Linear
Monotonically Increasing	Concave (downward)
Monotonically Decreasing	Convex (upward)
Unimodal	Concave then convex
Bathtub	Convex then concave

2.5. Akaike's Information Criteria (AIC)

Introduced by Hirotugu Akaike in 1973, AIC aims to select a model that provides the optimal balance between goodness-of-fit and model complexity. Goodness-of-fit is measured by how well a model explains the observed data, represented by the likelihood value. Mathematically, AIC is formulated as follows.

$$AIC = -2logL + 2k \tag{14}$$

Where L is the maximum value of the likelihood function and k is the number of parameters in the model. The addition of $2k$ serves as a penalty term to prevent overfitting due to overly complex models. In practice, AIC is seen not only as a model selection tool but also as an effective prediction evaluation method, especially in large-sample contexts [12].

3. Methods

3.1. Construction of the Chen-Burr XII Distribution

This section presents the construction of the proposed model using the Transformed-Transformer $(T - X)$ method. This approach provides a flexible framework for generating new probability distributions by combining two existing distributions, referred to as the transformed distribution and the transformer distribution. The Chen-Burr XII distribution is constructed using the $T - X$ method, where T a random variable follows the Chen distribution with a PDF defined for $t \geq 0$ and parameters $\lambda, \beta > 0$, as given in Eq. (4). Meanwhile, X is a random variable that follows the Burr XII distribution with a CDF characterized by parameters $c, k > 0$, as presented in Eq. (7). Since the random variable T, which follows the Chen distribution, is defined over the support $T \in [0, \infty)$, the transformation function $W(F(x))$ used in the $T - X$ method must lie within the domain $[a, \infty)$.

In this study, the transformation function is defined as $W\big(F(x)\big) = -\log\big(1 - F(x)\big)$, where $F(x)$ denotes the cumulative distribution function (CDF) of the Burr Type XII distribution, as given in Eq. (7). From Eq. (7), it follows that $1 - F(x) = (1 + x^c)^{-k}$. Substituting this expression into the function $W(F(x))$ yields:

$$W\big(F(x)\big) = k\log(1 + x^c) \tag{15}$$

Eq. (15) is strictly increasing and has a positive transformation for $x > 0$, ensuring that the support of the transformed variable is compatible with the domain of the Chen distribution. Eq. (15) has a logarithmic form; therefore, the upper bound of the integral in the T–X method produces a cumulative distribution function $G(x)$that is more tractable for analytical derivation. The CDF of the proposed distribution can be obtained by substituting $W(F(x))$ from Eq. (15) and $r(t)$ the PDF of Chen distribution given in Eq. (4) into Eq. (2), which yields:

$$G(x) = \int_0^{klog\,(1+x^c)} \lambda\beta t^{\beta-1} \exp\left[t^\beta + \lambda\left(1 - e^{t^\beta}\right)\right] dt \tag{16}$$

Note that Eq. (16) represents the integral of the Chen probability density function with upper limit $t = k\log(1 + x^c)$. Therefore, by the Fundamental Theorem of Calculus, this integral can be expressed as the cumulative distribution function (CDF) of the Chen distribution evaluated at the transformed upper bound. Let $R(t)$ denote the CDF of the Chen distribution. Since the lower bound of the integral is also zero and $R(0) = 0$, it follows that

$$G(x) = R(k\log(1 + x^c)) \qquad (17)$$

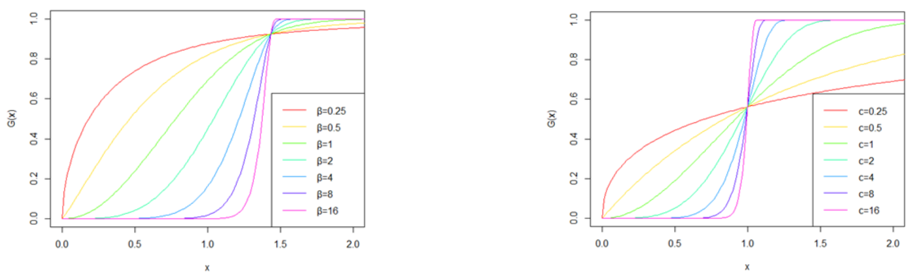
Eq. (17) provides a convenient form for deriving the explicit expression of the Chen-Burr XII distribution, which will be further discussed in the next section.

3.2. Properties of the Chen-Burr XII Distribution

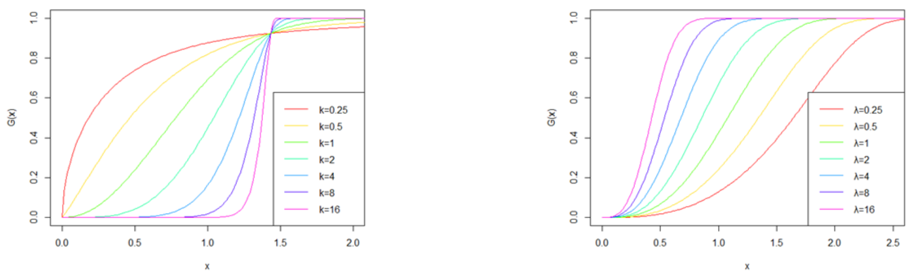
Using Eq. (17), the cumulative distribution function (CDF) of the Chen-Burr XII for $x > 0$ and $\lambda, \beta, c, k > 0$ is given by:

$$G(x) = 1 - \exp\left\{\lambda\left[1 - \exp\left(\left[\log\left((1 + x^c)^k\right)\right]^\beta\right)\right]\right\} \qquad (18)$$

Figure 3. illustrates the behavior of the cumulative distribution function under varying parameter values, with three parameters held constant.



(a) $\lambda = 1.5; c = 1.5; k = 1.5;\ \beta$ varies (b) $\lambda = 1.5; \beta = 1.5; k = 1.5;\ c$ varies



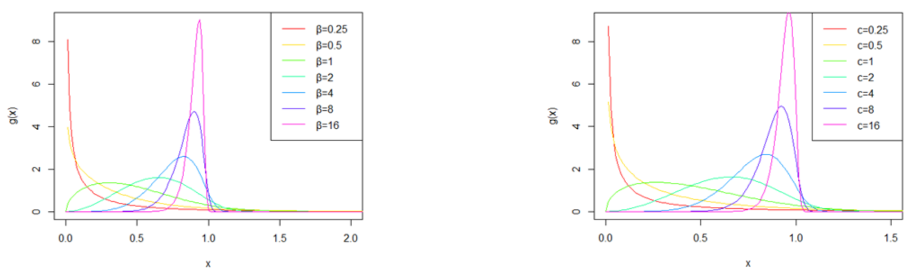
(c) $\lambda = 1.5; c = 1.5; \beta = 1.5;\ k$ varies (d) $\beta = 1.5; c = 1.5; k = 1.5;\ \lambda$ varies

Figure 3. Plots of Chen-Burr XII CDF

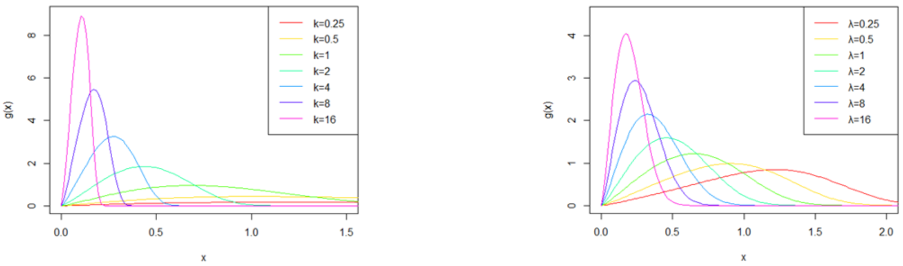
The probability density function (PDF) is obtained by differentiating the CDF of the Chen-Burr XII distribution with respect to x, resulting in:

$$g(x) = \lambda\beta ck\frac{x^{c-1}}{1+x^c}[k\log(1 + x^c)]^{\beta-1} \cdot \exp\left[[k\log(1 + x^c)]^\beta + \lambda\left(1 - e^{[k\log(1+x^c)]^\beta}\right)\right] \qquad (19)$$

Figure 4. illustrates the behavior of the probability density function under varying parameter values, with three parameters held constant, demonstrating the flexibility of the distribution.



(a) $\lambda = 1.5; c = 1.5; k = 1.5;\ \beta$ varies (b) $\lambda = 1.5; \beta = 1.5; k = 1.5;\ c$ varies



(c) $\lambda = 1.5; c = 1.5; \beta = 1.5;\ k$ varies (d) $\beta = 1.5; c = 1.5; k = 1.5;\ \lambda$ varies

Figure 4. Plots of Chen-Burr XII PDF

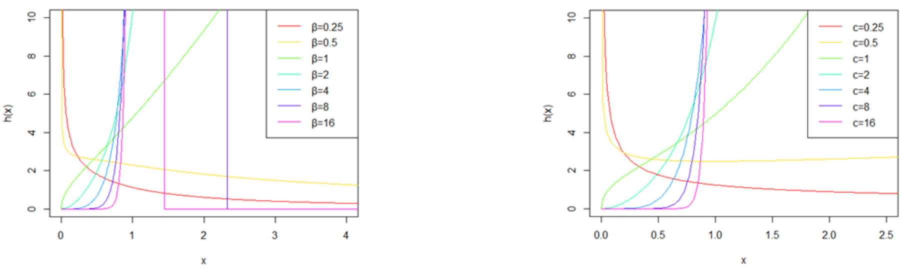
The survival function (SF) is defined as $S(x) = 1 - G(x)$. Thus, for $x > 0$ and $\lambda, \beta, c, k > 0$, it is given by:

$$S(x) = \exp\{\lambda[1 - \exp([k \log(1 + x^c)]^\beta)]\} \tag{20}$$

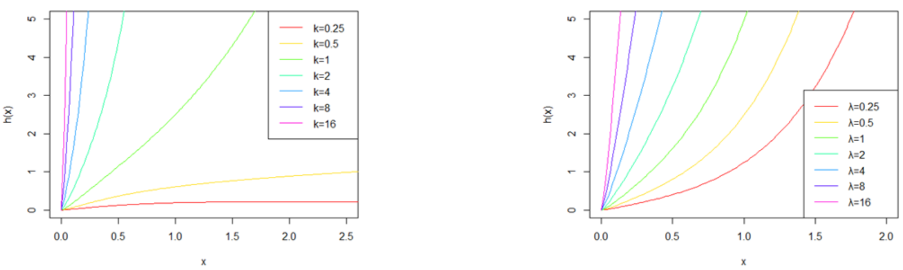
The hazard function (HF) is defined as $h(x) = \frac{g(x)}{S(x)}$, which yields:

$$h(x) = \lambda\beta c k x^{c-1} \cdot \frac{[k \log(1+x^c)]^{\beta-1}}{1+x^c} \cdot \exp[(k \log(1 + x^c))^\beta] \tag{21}$$

Figure 5. illustrates the hazard function under varying values of one parameter while the remaining parameters are held constant, resulting in monotonic hazard shapes, including increasing and decreasing patterns.



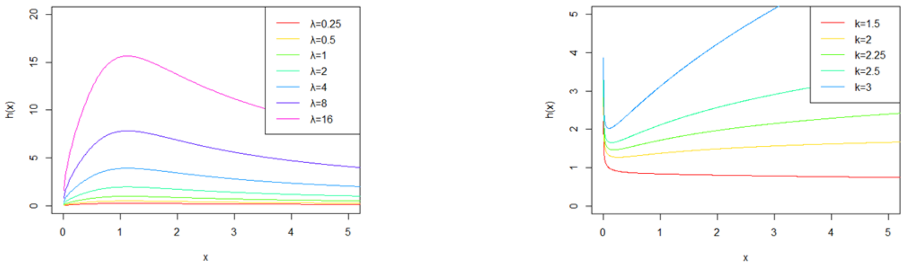
(a) $\lambda = 1.5; c = 1.5; k = 1.5;\ \beta$ varies (b) $\lambda = 1.5; \beta = 1.5; k = 1.5;\ c$ varies



(c) $\lambda = 1.5; c = 1.5; \beta = 1.5;\ k$ varies (d) $\beta = 1.5; c = 1.5; k = 1.5;\ \lambda$ varies

Figure 5. Plots of Chen-Burr XII HF

Figure 6. presents further exploration of the parameter space, demonstrating the ability of the Chen-Burr XII distribution to capture more complex hazard shapes, such as unimodal and bathtub patterns.



(a) c = 2.5; k = 0.5; β = 0.6; λ varies (b) $\lambda = 0.5; c = 1, \beta = 0.8;\ k$ varies

Figure 6. Variation plots of Chen-Burr XII HF

Another characteristic of the distribution analyzed is the *r*-th moment, where $I_{j,r} = \int_0^\infty \exp\left(\frac{t^{\frac{1}{\beta}}}{k}\left(\frac{r}{c} - j\right) + t - \lambda e^t\right) dt$ and is given as follows:

$$E(X^r) = \lambda \sum_{j=0}^{\infty} \binom{r/c}{j} (-1)^j \cdot I_{j,r} \tag{22}$$

The first and second moments correspond to the expectation and variance, respectively, as follows:

$$E(X) = \lambda \sum_{j=0}^{\infty} \binom{1/c}{j} (-1)^j \cdot I_{j,1} \tag{23}$$

And

$$Var(X) = \lambda \sum_{j=0}^{\infty} \binom{2/c}{j} (-1)^j \cdot I_{j,2} - \left[\lambda \sum_{j=0}^{\infty} \binom{1/c}{j} (-1)^j \cdot I_{j,1}\right]^2 \tag{24}$$

The third and fourth moments represent the coefficients of skewness and kurtosis, respectively:

$$E(X^3) = \lambda \sum_{j=0}^{\infty} \binom{3/c}{j} (-1)^j \cdot I_{j,3} \tag{25}$$

And

$$E(X^4) = \lambda \sum_{j=0}^{\infty} \binom{4/c}{j} (-1)^j \cdot I_{j,4} \tag{26}$$

The moment generating function (MGF) is derived using $g(x)$ the PDF of the Chen-Burr XII distribution. By expressing e^{tx} through Taylor series expansion, $e^{tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n$. The MGF is obtained as follows:

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\lambda \sum_{j=0}^{\infty} \binom{n/c}{j} (-1)^j I_{j,n}\right] \tag{27}$$

3.3. Parameter Estimation of the Chen-Burr XII Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Chen-Burr XII distribution with parameter vector $\boldsymbol{\phi} = (\lambda, \beta, c, k)$ and let $x_1, x_2, \ldots, x_n$ denote the observed values. If $f(x)$ is the corresponding probability density function and let $u_i = k \log(1 + x_i^c)$ then the likelihood function and the log-likelihood function respectively, given by:

$$L(\boldsymbol{\phi}) = f(x_1, x_2, \ldots, x_n; \boldsymbol{\phi})$$

$$L(\boldsymbol{\phi}) = \prod_{i=1}^{n} \left[\lambda \beta c k \frac{x_i^{c-1}}{1 + x_i^c} \cdot [k \log(1 + x_i^c)]^{\beta - 1} \cdot \exp\left[(k \log(1 + x_i^c))^\beta + \lambda\left(1 - e^{[k \log(1 + x_i^c)]^\beta}\right)\right]\right]$$

And

$$\ell(\boldsymbol{\phi}) = n\log\lambda + n\log\beta + n\log c + n\log k + (c-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\log(1+x_i^c) + (\beta-1)\sum_{i=1}^{n}\log u_i + \sum_{i=1}^{n} u_i^{\beta} + \lambda\sum_{i=1}^{n}\left(1-e^{u_i^{\beta}}\right) \quad (28)$$

The partial derivatives of Eq. (28) with respect to each parameter are expressed as follows:

$$\frac{\partial\ell}{\partial\lambda} = \hat{\lambda} = \frac{n}{\sum_{i=1}^{n}\left(e^{[k\log(1+x_i^c)]^{\beta}} - 1\right)} \quad (29)$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log u_i + \sum_{i=1}^{n} u_i^{\beta}\log u_i - \lambda\sum_{i=1}^{n}\left(e^{u_i^{\beta}} u_i^{\beta}\log u_i\right) = 0 \quad (30)$$

$$\frac{\partial\ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{x_i^c\log x_i}{1+x_i^c} + (\beta-1)\sum_{i=1}^{n}\frac{x_i^c\log x_i}{(1+x_i^c)\log(1+x_i^c)} + \beta\sum_{i=1}^{n}\frac{kx_i^c\log x_i}{1+x_i^c}[k\log(1+x_i^c)]^{\beta-1} - \lambda\beta\sum_{i=1}^{n} e^{[k\log(1+x_i^c)]^{\beta}}\frac{kx_i^c\log x_i}{1+x_i^c} \cdot [k\log(1+x_i^c)]^{\beta-1} = 0 \quad (31)$$

$$\frac{\partial\ell}{\partial k} = \frac{\beta}{k}\left[n + \sum_{i=1}^{n}[k\log(1+x_i^c)]^{\beta} - \lambda\sum_{i=1}^{n} e^{[k\log(1+x_i^c)]^{\beta}}[k\log(1+x_i^c)]^{\beta}\right] = 0 \quad (32)$$

Eq. (29) provides the maximum likelihood estimator for λ. However, Eqs. (30)–(32) do not admit closed-form solutions and therefore require numerical methods, such as the Newton–Raphson algorithm, for parameter estimation.

4. Results and Discussion

4.1. Data on Waiting Time to Death of Gastric Cancer Patients

The dataset consists of survival times (in days) for 42 gastric cancer patients who received chemotherapy treatment. These data were originally reported in 1928 by the Gastrointestinal Tumor Study Group [13].

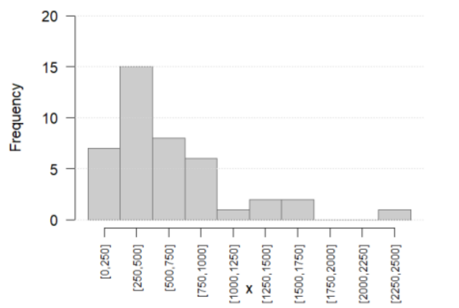


Figure 7. Histogram of Waiting Time Data for Death of Gastric Cancer Patients

The shape of the hazard function for the gastric cancer waiting time data is examined using the Total Time-on-Test (TTT) plot. The TTT curve is generated using the ‘TTT’ function from the “AdequacyModel” package in RStudio. In addition to the TTT plot, the hazard shape is further assessed using the *EstimationTools* package via the function *TTT_hazard_shape()*. The results from both approaches consistently confirm a unimodal (upside-down bathtub) hazard pattern, as shown in the figures below.

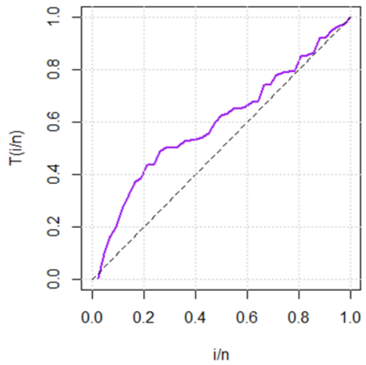


Figure 8. TTT-*plot* Data on Waiting Times for Death in Gastric Cancer Patients

```
> TTT_hazard_shape(Data_Lambung~1)$hazard_type
[1] "Unimodal"
```

Figure 9. Hazard Data Output for Gastric Cancer Death Waiting Time

Therefore, the Chen-Burr XII distribution can be considered suitable for modeling the waiting time to death of gastric cancer patients exhibiting a unimodal hazard pattern.

4.2. Model Fit Test

Parameter estimation for each distribution is initially carried out using the Maximum Likelihood Estimation (MLE) method, followed by goodness-of-fit evaluation using the Kolmogorov–Smirnov (KS) test. The results for the proposed Chen-Burr XII distribution, as well as the Chen and Burr Type XII distributions as comparative models, are summarized in Table 2.

Table 2. Parameter Estimates and Kolmogorov–Smirnov Goodness-of-Fit Results for the Chen, Burr Type XII, and Chen-Burr XII Distributions

Model Distribution	**MLE**				**Kolmogorov-Smirnov**
	$\hat{\lambda}$	$\hat{\beta}$	$\hat{c}$	$\hat{k}$	***p-value***
Chen	0.08675	0.46384	–	–	< 0.12567 (0.20985)
Burr Type XII	–	–	0.73675	0.54032	> 0.280484 (0.20985)
Chen-Burr XII	0.01581	0.72287	1.15230	3.09888	< 0.19791 (0.20985)

Based on the Kolmogorov-Smirnov test results presented in Table 2, the waiting time to death data for gastric cancer patients are not adequately fitted by the Burr Type XII distribution, whereas the Chen and Chen-Burr XII distributions provide a better fit to the observed data. To further evaluate model performance, a graphical approach is conducted. The goodness-of-fit is assessed by comparing the probability density function and cumulative distribution function of the parametric models with the empirical data, respectively, by:

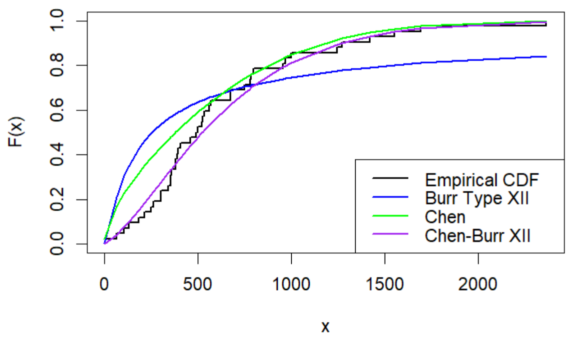


Figure 10. Comparison of Empirical CDF Curves of Waiting Time for Death in Gastric Cancer Patients with Burr Type XII, Chen, and Chen-Burr XII Distributions

The CDF curves of the Chen-Burr XII distribution (purple) and the Chen distribution (green) are closest to the empirical CDF curve (black) across the entire data range. Meanwhile, the Burr Type XII distribution (blue line) appears less capable of describing the data pattern because it is far below the empirical curve. The following are the results of the comparison of the three PDF curves against the empirical data.

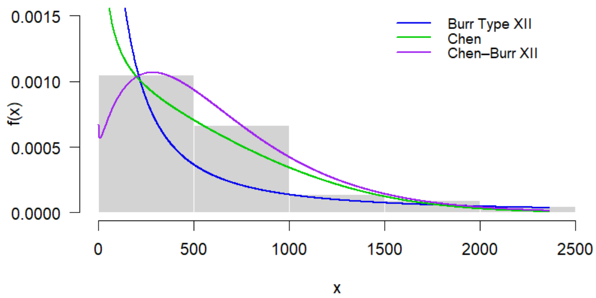


Figure 11. Comparison of Empirical PDF Curves of Waiting Time for Death in Gastric Cancer Patients with Burr Type XII, Chen, and Chen-Burr XII Distributions

From Figure 11, the Chen-Burr XII distribution curve (purple) best fits the histogram pattern of the data, especially at the peak and right tail of the data. Based on the probability density and cumulative distribution function plots, as well as the Kolmogorov–Smirnov (K–S) test results, the Chen and Chen-Burr XII distributions provide a better fit to the gastric cancer data than the Burr Type XII distribution. To determine the most appropriate model, the Akaike Information Criterion (AIC) values of the competing distributions are compared, with the model corresponding to the lowest AIC value selected as the best-fitting model. The comparison results are presented in Table 3.

Table 3. Comparison of AIC Values

Model Distribution	AIC Value
Chen	244.61315
Chen-Burr XII	239.29765

The results indicate that the Chen–Burr XII distribution provides a better fit than the Chen distribution in modeling the waiting time to death data of 42 gastric cancer patients.

5. Conclusion

This study introduces the Chen–Burr XII distribution using the Transformed–Transformer (T–X) method, where the Chen distribution is used as the transformed distribution and the Burr Type XII distribution acts as the transformer. This construction combines the probability density function of the Chen distribution with a transformation based on the cumulative distribution function of the Burr Type XII distribution, resulting in a more flexible model particularly in capturing more complex hazard patterns.

The proposed distribution is a continuous distribution with four positive parameters and has closed-form expressions for its main functions. It shows greater flexibility in modeling survival data, as its probability density function can take decreasing and unimodal shapes, while its hazard function can represent increasing, decreasing, unimodal, and bathtub-shaped patterns. These results help overcome the limitations of the original Chen distribution in modeling more complex hazard behaviors.

Parameter estimation is carried out using Maximum Likelihood Estimation, which requires numerical methods such as the Newton–Raphson algorithm since closed-form solutions are not available. Overall, the Chen-Burr XII distribution provides a flexible and useful model for lifetime data and contributes to survival analysis.

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